Problem Set: Human Capital and the Lucas Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Unless otherwise stated, assume all variables are strictly positive and all parameters lie in economically meaningful ranges. The problem set is based on the augmented Solow model with human capital and the Lucas human-capital accumulation model.

Question 1: Human Capital, Conditional Income Gaps, and Augmented Solow Dynamics [Total: 45 marks]

A major empirical fact in growth economics is that cross-country income differences are strongly correlated with schooling, but measured education alone does not explain all international income gaps. Consider an augmented Solow–Swan economy in which human capital enters production through effective labour:

$$Y_t = K_t^\alpha [A_t H_t]^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where

$$H_t = L_t G(x), \qquad G'(x) > 0, \qquad G''(x) > 0.$$

Here x is average years of schooling, assumed constant over time for a given country. Technology and population evolve as

$$A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t,$$

and capital accumulation is

$$K_{t+1} = (1-\delta)K_t + sY_t, \qquad 0 < s < 1, \qquad 0 < \delta < 1.$$

Define

$$k_t \equiv \frac{K_t}{A_t L_t G(x)}, \qquad D \equiv (1+n)(1+g), \qquad b \equiv D - (1-\delta).$$

(a) Derive the intensive-form production function

$$\widetilde{y}_t \equiv \frac{Y_t}{A_t L_t G(x)}$$

and the exact law of motion for k_t. Then derive $k_{t+1} - k_t$ and interpret the break-even investment term. **[10 marks]**

(b) Solve for the positive steady-state capital stock k^* and output per worker $(Y_t/L_t)^*$. Assume $G(x) = e^{\eta x}$, where $\eta > 0$. Derive the predicted relative output per worker of country i relative to country j:

$$\frac{(Y/L)_i^*}{(Y/L)_j^*}.$$

Explain why this is a conditional-convergence prediction. **[12 marks]**

(c) Suppose country R is a rich economy and country P is a poor economy. They share

$$\alpha = \frac{1}{3}, \qquad \eta = 0.08, \qquad g = 0.02, \qquad \delta = 0.05,$$

but differ in saving, population growth, and schooling:

$$s_R = 0.28, \quad n_R = 0.01, \quad x_R = 12,$$

$$s_P = 0.12, \quad n_P = 0.025, \quad x_P = 7.$$

Using the exact formula for b, compute

$$\frac{(Y/L)_R^*}{(Y/L)_P^*}.$$

Decompose the ratio into the education component and the capital-deepening/demography component. **[10 marks]**

(d) Prove local convergence to the steady state. Derive the convergence coefficient

$$\lambda = \mathcal{G}'(k^*).$$

Explain why countries with the same structural parameters but lower initial k_0 grow faster during transition. **[8 marks]**

(e) Suppose schooling changes over time. Derive the growth-accounting equation for output per worker:

$$\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1-\alpha)\Delta \ln A_t + (1-\alpha)\Delta \ln G(x_t).$$

If $G(x) = e^{\eta x}$, rewrite the human-capital contribution in terms of Δx_t. **[5 marks]**

Question 2: The Lucas Human-Capital Model and Endogenous Growth [Total: 55 marks]

Consider the Lucas economy. A representative agent has one unit of time each period. The fraction φ_t is used in goods production, while $1-\varphi_t$ is used in education. Preferences are

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta}-1}{1-\theta}, \qquad 0<\beta<1, \qquad \theta>1.$$

Goods production is

$$Y_t = K_t^{\alpha}(\varphi_t H_t)^{1-\alpha}, \qquad 0<\alpha<1,$$

and human capital evolves according to

$$H_{t+1} = B(1-\varphi_t)H_t, \qquad B>0.$$

The resource constraint is

$$C_t + K_{t+1} = K_t^{\alpha}(\varphi_t H_t)^{1-\alpha} + (1-\delta)K_t, \qquad 0<\delta<1.$$

Initial stocks $K_0>0$ and $H_0>0$ are given.

(a) Set up the social planner's Lagrangian. Derive the first-order conditions with respect to φ_t, H_{t+1}, and K_{t+1}. Write the FOCs in terms of Y_t. **[15 marks]**

(b) Eliminate the human-capital multiplier to derive

$$\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \beta B\left[\frac{Y_{t+1}}{Y_t}\frac{H_t}{H_{t+1}}\frac{\varphi_t}{\varphi_{t+1}}\right].$$

Then derive the physical-capital Euler equation

$$\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \beta\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].$$

[13 marks]

(c) Define a balanced-growth path as an allocation in which C_t, K_t, Y_t, and H_t all grow at the common constant net rate g, while $\varphi_t=\varphi^*$ is constant. Derive

$$1+g = (\beta B)^{1/\theta},$$

and

$$\varphi^* = 1 - \frac{(\beta B)^{1/\theta}}{B}.$$

State the conditions for positive endogenous growth and an interior time allocation $0<\varphi^*<1$. **[10 marks]**

(d) Along the balanced-growth path, derive the output-capital ratio Y/K, the capital-output ratio K/Y, and the consumption-output ratio C/Y. Show that

$$\frac{Y}{K} = \frac{B-1+\delta}{\alpha}.$$

Then derive the condition under which $C/Y>0$. **[10 marks]**

(e) Let

$$\beta=0.96, \qquad \theta=2, \qquad \alpha=\frac{1}{3}, \qquad \delta=0.05.$$

Compare two education technologies:

$$B_0 = 1.05, \qquad B_1 = 1.10.$$

For each case, compute g, φ^*, Y/K, K/Y, and C/Y. Interpret the effect of a higher return to education. **[7 marks]**

Detailed Solutions

Solution to Question 1 [45 marks]

Part (a) [10 marks]

The production function is

$$Y_t = K_t^\alpha [A_t L_t G(x)]^{1-\alpha}.$$

Divide by $A_t L_t G(x)$:

$$\widetilde{y}_t = \frac{Y_t}{A_t L_t G(x)} = \frac{K_t^\alpha [A_t L_t G(x)]^{1-\alpha}}{A_t L_t G(x)}.$$

Hence,

$$\widetilde{y}_t = K_t^\alpha [A_t L_t G(x)]^{-\alpha} = \left(\frac{K_t}{A_t L_t G(x)}\right)^\alpha.$$

Therefore,

$$\boxed{\widetilde{y}_t = k_t^\alpha.}$$

Capital accumulation is

$$K_{t+1} = (1-\delta)K_t + sY_t.$$

Divide by $A_t L_t G(x)$:

$$\frac{K_{t+1}}{A_t L_t G(x)} = (1-\delta)k_t + sk_t^\alpha.$$

Since x is constant over time for a given country,

$$A_{t+1}L_{t+1}G(x) = (1+g)(1+n)A_t L_t G(x) = DA_t L_t G(x).$$

Therefore,

$$\frac{K_{t+1}}{A_t L_t G(x)} = Dk_{t+1}.$$

Thus,

$$Dk_{t+1} = (1-\delta)k_t + sk_t^\alpha.$$

The exact law of motion is

$$\boxed{k_{t+1} = \mathcal{G}(k_t) = \frac{(1-\delta)k_t + sk_t^\alpha}{D}.}$$

Subtracting k_t:

$$k_{t+1} - k_t = \frac{(1-\delta)k_t + sk_t^\alpha - Dk_t}{D}.$$

Because

$$b = D - (1-\delta),$$

we obtain

$$\boxed{k_{t+1} - k_t = \frac{sk_t^\alpha - bk_t}{D}.}$$

The term sk_t^α is actual investment per unit of effective human-capital-adjusted labour. The term bk_t is break-even investment: it is the investment required to keep k_t constant despite population growth, technological growth, their interaction, and depreciation.

Marking guide: intensive-form production, 3; transition equation, 4; $k_{t+1} - k_t$, 2; interpretation, 1.

Part (b) [12 marks]

A positive steady state satisfies

$$k_{t+1} = k_t = k^*.$$

From the transition equation,

$$k^* = \frac{(1-\delta)k^* + s(k^*)^\alpha}{D}.$$

Multiplying by D:

$$Dk^* = (1-\delta)k^* + s(k^*)^\alpha.$$

Therefore,

$$[D - (1-\delta)]\, k^* = s(k^*)^\alpha.$$

Since $b = D - (1-\delta)$,

$$bk^* = s(k^*)^\alpha.$$

For $k^* > 0$,

$$b(k^*)^{1-\alpha} = s.$$

Hence,

$$\boxed{k^* = \left(\frac{s}{b}\right)^{1/(1-\alpha)}.}$$

Output per worker is

$$\frac{Y_t}{L_t} = \frac{K_t^\alpha [A_t L_t G(x)]^{1-\alpha}}{L_t}.$$

Using

$$K_t = k_t A_t L_t G(x),$$

we get

$$\frac{Y_t}{L_t} = A_t G(x) k_t^\alpha.$$

Therefore, at the steady state,

$$\boxed{\left(\frac{Y_t}{L_t}\right)^* = A_t G(x)(k^*)^\alpha = A_t G(x)\left(\frac{s}{b}\right)^{\alpha/(1-\alpha)}.}$$

Now suppose

$$G(x) = e^{\eta x}.$$

For countries i and j evaluated at a common technology level A_t,

$$\frac{(Y/L)_i^*}{(Y/L)_j^*} = \frac{A_t e^{\eta x_i}(s_i/b_i)^{\alpha/(1-\alpha)}}{A_t e^{\eta x_j}(s_j/b_j)^{\alpha/(1-\alpha)}}.$$

Canceling A_t gives

$$\boxed{\frac{(Y/L)_i^*}{(Y/L)_j^*} = e^{\eta(x_i - x_j)}\left[\frac{s_i}{s_j}\frac{b_j}{b_i}\right]^{\alpha/(1-\alpha)}.}$$

Using the Solow approximation

$$b_i \approx n_i + g + \delta,$$

this becomes

$$\boxed{\frac{(Y/L)_i^*}{(Y/L)_j^*} = e^{\eta(x_i - x_j)}\left[\frac{s_i}{s_j}\frac{n_j+g+\delta}{n_i+g+\delta}\right]^{\alpha/(1-\alpha)}.}$$

This is a conditional-convergence prediction because long-run income depends on structural characteristics: saving, population growth, technology growth, depreciation, and schooling. Countries do not necessarily converge to the same income level; they converge to their own conditional steady states.

Marking guide: k^*, 3; output per worker, 3; relative-income formula, 4; conditional-convergence interpretation, 2.

Part (c) **[10 marks]**

For country R,

$$b_R = (1+n_R)(1+g) - (1-\delta).$$

Using

$$n_R = 0.01, \qquad g = 0.02, \qquad \delta = 0.05,$$

$$b_R = (1.01)(1.02) - 0.95 = 1.0302 - 0.95 = 0.0802.$$

For country P,

$$b_P = (1+n_P)(1+g) - (1-\delta).$$

Using

$$n_P = 0.025,$$

$$b_P = (1.025)(1.02) - 0.95 = 1.0455 - 0.95 = 0.0955.$$

The relative output formula is

$$\frac{(Y/L)_R^*}{(Y/L)_P^*} = e^{\eta(x_R - x_P)}\left[\frac{s_R}{s_P}\frac{b_P}{b_R}\right]^{\alpha/(1-\alpha)}.$$

Since

$$\alpha = \frac{1}{3}, \qquad \frac{\alpha}{1-\alpha} = \frac{1}{2},$$

$$\frac{(Y/L)^*_R}{(Y/L)^*_P} = e^{0.08(12-7)} \left[\frac{0.28}{0.12} \frac{0.0955}{0.0802} \right]^{1/2}.$$

The education component is

$$e^{0.08(5)} = e^{0.4} \approx 1.4918.$$

The capital-deepening/demography component is

$$\left[\frac{0.28}{0.12} \frac{0.0955}{0.0802} \right]^{1/2} = [2.3333 \times 1.1908]^{1/2}.$$

Thus,

$$[2.7785]^{1/2} \approx 1.6669.$$

Therefore,

$$\frac{(Y/L)^*_R}{(Y/L)^*_P} \approx 1.4918 \times 1.6669 \approx 2.487.$$

Hence,

$$\boxed{\frac{(Y/L)^*_R}{(Y/L)^*_P} \approx 2.49.}$$

The model predicts that country R has about 2.49 times the steady-state output per worker of country P. Around 1.49 of this ratio is due to schooling, while around 1.67 is due to higher saving and lower capital dilution from population growth.

Marking guide: b_R and b_P, 2; formula, 2; education component, 2; capital-deepening component, 2; final ratio and interpretation, 2.

Part (d) **[8 marks]**

The transition function is

$$\mathcal{G}(k) = \frac{(1-\delta)k + sk^\alpha}{D}.$$

Differentiate:

$$\mathcal{G}'(k) = \frac{(1-\delta) + s\alpha k^{\alpha-1}}{D}.$$

At the steady state,

$$s(k^*)^{\alpha-1} = b.$$

Therefore,

$$\boxed{\lambda = \mathcal{G}'(k^*) = \frac{(1-\delta) + \alpha b}{D}.}$$

To show convergence, note that

$$k_{t+1} - k_t = \frac{k_t(sk_t^{\alpha-1} - b)}{D}.$$

Since $0 < \alpha < 1$, $\alpha - 1 < 0$, so $k^{\alpha-1}$ is strictly decreasing. At k^*,

$$s(k^*)^{\alpha-1} = b.$$

If $k_t < k^*$, then

$$k_t^{\alpha-1} > (k^*)^{\alpha-1},$$

so

$$sk_t^{\alpha-1} > b,$$

and therefore

$$k_{t+1} - k_t > 0.$$

If $k_t > k^*$, then

$$sk_t^{\alpha-1} < b,$$

so

$$k_{t+1} - k_t < 0.$$

Thus, k_t converges to k^*.

Conditional convergence means that, holding $(s, n, g, \delta, x, \alpha)$ fixed, an economy below its steady state has actual investment greater than break-even investment. Therefore, its capital per effective worker rises, and output per worker grows faster during transition. A poorer economy catches up only conditional on having the same long-run determinants.
Marking guide: derivative and λ, 3; convergence proof, 3; economic interpretation, 2.

Part (e) **[5 marks]**

Output per worker is

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^{\alpha} A_t^{1-\alpha} G(x_t)^{1-\alpha}.$$

Taking logs:

$$\ln(Y_t/L_t) = \alpha \ln(K_t/L_t) + (1-\alpha)\ln A_t + (1-\alpha)\ln G(x_t).$$

Taking log differences:

$$\boxed{\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1-\alpha)\Delta \ln A_t + (1-\alpha)\Delta \ln G(x_t).}$$

If

$$G(x_t) = e^{\eta x_t},$$

then

$$\ln G(x_t) = \eta x_t.$$

Therefore,

$$\Delta \ln G(x_t) = \eta \Delta x_t.$$

Hence,

$$\boxed{\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1-\alpha)\Delta \ln A_t + (1-\alpha)\eta \Delta x_t.}$$

The last term is the direct contribution of rising schooling to output-per-worker growth.
Marking guide: log-linear decomposition, 3; schooling contribution under $G(x) = e^{\eta x}$, 2.

Solution to Question 2 [55 marks]

Part (a) [15 marks]

The planner chooses sequences

$$\{C_t, K_{t+1}, H_{t+1}, \varphi_t\}_{t=0}^{\infty}$$

to maximise

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta}$$

subject to

$$C_t + K_{t+1} = K_t^{\alpha}(\varphi_t H_t)^{1-\alpha} + (1-\delta)K_t,$$

and

$$H_{t+1} = B(1-\varphi_t)H_t.$$

Using the resource constraint to substitute for consumption,

$$C_t = K_t^{\alpha}(\varphi_t H_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}.$$

The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\theta} - 1}{1-\theta} + \lambda_t \left[B(1-\varphi_t)H_t - H_{t+1}\right] \right\}.$$

The derivative of output with respect to φ_t is

$$\frac{\partial Y_t}{\partial \varphi_t} = (1-\alpha)\frac{Y_t}{\varphi_t}.$$

The first-order condition with respect to φ_t is

$$C_t^{-\theta}(1-\alpha)\frac{Y_t}{\varphi_t} - \lambda_t B H_t = 0.$$

Therefore,

$$\boxed{C_t^{-\theta}\left[\frac{(1-\alpha)Y_t}{\varphi_t}\right] = \lambda_t B H_t.}$$

The variable H_{t+1} appears negatively in the time-t human-capital constraint and positively in next period's production and human-capital law. The derivative of Y_{t+1} with respect to H_{t+1} is

$$\frac{\partial Y_{t+1}}{\partial H_{t+1}} = (1-\alpha)\frac{Y_{t+1}}{H_{t+1}}.$$

The first-order condition with respect to H_{t+1} is

$$-\lambda_t + \beta\left\{C_{t+1}^{-\theta}\left[\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}\right] + \lambda_{t+1}B(1-\varphi_{t+1})\right\} = 0.$$

Thus,

$$\boxed{\lambda_t = \beta\left\{C_{t+1}^{-\theta}\left[\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}\right] + \lambda_{t+1}B(1-\varphi_{t+1})\right\}.}$$

The derivative of Y_{t+1} with respect to K_{t+1} is

$$\frac{\partial Y_{t+1}}{\partial K_{t+1}} = \alpha\frac{Y_{t+1}}{K_{t+1}}.$$

The first-order condition with respect to K_{t+1} is

$$-C_t^{-\theta} + \beta C_{t+1}^{-\theta}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right] = 0.$$

Hence,

$$\boxed{C_t^{-\theta} = \beta C_{t+1}^{-\theta}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].}$$

Marking guide: Lagrangian, 3; FOC for φ_t, 4; FOC for H_{t+1}, 4; FOC for K_{t+1}, 4.

Part (b) **[13 marks]**

From the FOC for φ_t,

$$C_t^{-\theta}\left[\frac{(1-\alpha)Y_t}{\varphi_t}\right] = \lambda_t B H_t.$$

Therefore,

$$\lambda_t = \frac{C_t^{-\theta}(1-\alpha)Y_t}{\varphi_t B H_t}.$$

Similarly,

$$\lambda_{t+1} = \frac{C_{t+1}^{-\theta}(1-\alpha)Y_{t+1}}{\varphi_{t+1} B H_{t+1}}.$$

Substitute these expressions into the FOC for H_{t+1}:

$$\lambda_t = \beta\left\{C_{t+1}^{-\theta}\left[\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}\right] + \lambda_{t+1}B(1-\varphi_{t+1})\right\}.$$

This gives

$$\frac{C_t^{-\theta}(1-\alpha)Y_t}{\varphi_t B H_t} = \beta\left\{C_{t+1}^{-\theta}\frac{(1-\alpha)Y_{t+1}}{H_{t+1}} + \frac{C_{t+1}^{-\theta}(1-\alpha)Y_{t+1}}{\varphi_{t+1} B H_{t+1}}B(1-\varphi_{t+1})\right\}.$$

Simplify the term in braces:

$$C_{t+1}^{-\theta}\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}\left[1 + \frac{1-\varphi_{t+1}}{\varphi_{t+1}}\right].$$

Since

$$1 + \frac{1-\varphi_{t+1}}{\varphi_{t+1}} = \frac{1}{\varphi_{t+1}},$$

we obtain

$$\frac{C_t^{-\theta}(1-\alpha)Y_t}{\varphi_t B H_t} = \beta C_{t+1}^{-\theta}\frac{(1-\alpha)Y_{t+1}}{H_{t+1}\varphi_{t+1}}.$$

Canceling $(1-\alpha)$ and rearranging:

$$\frac{C_t^{-\theta}}{C_{t+1}^{-\theta}} = \beta B\frac{Y_{t+1}}{Y_t}\frac{H_t}{H_{t+1}}\frac{\varphi_t}{\varphi_{t+1}}.$$

Because

$$\frac{C_t^{-\theta}}{C_{t+1}^{-\theta}} = \left(\frac{C_{t+1}}{C_t}\right)^{\theta},$$

we get

$$\boxed{\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \beta B\left[\frac{Y_{t+1}}{Y_t}\frac{H_t}{H_{t+1}}\frac{\varphi_t}{\varphi_{t+1}}\right].}$$

From the FOC for K_{t+1},

$$C_t^{-\theta} = \beta C_{t+1}^{-\theta}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].$$

Divide both sides by $C_{t+1}^{-\theta}$:

$$\frac{C_t^{-\theta}}{C_{t+1}^{-\theta}} = \beta\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].$$

Hence,

$$\boxed{\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \beta\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].}$$

The first Euler equation governs the return to education. The second governs the return to physical capital. Along an optimal path, the planner equalises the utility-adjusted returns across these two accumulation margins.

Marking guide: expression for λ_t, 3; substitution into H_{t+1} FOC, 5; education Euler equation, 3; physical-capital Euler equation, 2.

Part (c) **[10 marks]**

On a balanced-growth path, suppose

$$\frac{C_{t+1}}{C_t} = \frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \frac{H_{t+1}}{H_t} = 1 + g,$$

and

$$\varphi_t = \varphi_{t+1} = \varphi^*.$$

Start from the education Euler equation:

$$\left(\frac{C_{t+1}}{C_t}\right)^\theta = \beta B \left[\frac{Y_{t+1}}{Y_t} \frac{H_t}{H_{t+1}} \frac{\varphi_t}{\varphi_{t+1}}\right].$$

Substitute the BGP restrictions:

$$(1+g)^\theta = \beta B \left[(1+g)\frac{1}{1+g} \times 1\right].$$

Therefore,

$$(1+g)^\theta = \beta B.$$

Hence,

$$\boxed{1 + g = (\beta B)^{1/\theta}.}$$

So the net growth rate is

$$\boxed{g = (\beta B)^{1/\theta} - 1.}$$

Now use the human-capital accumulation equation:

$$H_{t+1} = B(1 - \varphi_t)H_t.$$

On the balanced-growth path,

$$\frac{H_{t+1}}{H_t} = 1 + g.$$

Thus,

$$1 + g = B(1 - \varphi^*).$$

Substitute

$$1 + g = (\beta B)^{1/\theta}.$$

Then

$$(\beta B)^{1/\theta} = B(1 - \varphi^*).$$

Solving for φ^*:

$$1 - \varphi^* = \frac{(\beta B)^{1/\theta}}{B}.$$

Hence,

$$\boxed{\varphi^* = 1 - \frac{(\beta B)^{1/\theta}}{B}.}$$

Positive endogenous growth requires

$$g > 0.$$

Because

$$g = (\beta B)^{1/\theta} - 1,$$

positive growth requires

$$\boxed{\beta B > 1.}$$

An interior time allocation requires

$$0 < \varphi^* < 1.$$

Since

$$\varphi^* = 1 - \frac{(\beta B)^{1/\theta}}{B},$$

the condition $\varphi^* < 1$ requires

$$\frac{(\beta B)^{1/\theta}}{B} > 0,$$

which holds automatically. The condition $\varphi^* > 0$ requires

$$\frac{(\beta B)^{1/\theta}}{B} < 1.$$

Thus,

$$(\beta B)^{1/\theta} < B.$$

Equivalently,

$$\boxed{\beta B < B^\theta.}$$

When $\theta > 1$, this can be written as

$$\boxed{\beta < B^{\theta-1}.}$$

Marking guide: use of BGP in education Euler, 3; derivation of g, 2; derivation of φ^*, 3; parameter restrictions, 2.

Part (d) **[10 marks]**

The physical-capital Euler equation is

$$\left(\frac{C_{t+1}}{C_t}\right)^\theta = \beta\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)\right].$$

On the balanced-growth path,

$$\frac{C_{t+1}}{C_t} = 1+g,$$

and

$$\frac{Y_{t+1}}{K_{t+1}} = \frac{Y}{K}.$$

Thus,

$$(1+g)^\theta = \beta\left[\alpha\frac{Y}{K} + (1-\delta)\right].$$

From part (c),

$$(1+g)^\theta = \beta B.$$

Therefore,

$$\beta B = \beta\left[\alpha\frac{Y}{K} + (1-\delta)\right].$$

Divide by β:

$$B = \alpha\frac{Y}{K} + (1-\delta).$$

Hence,

$$\alpha\frac{Y}{K} = B - 1 + \delta.$$

Therefore,

$$\boxed{\frac{Y}{K} = \frac{B-1+\delta}{\alpha}.}$$

The capital-output ratio is the reciprocal:

$$\boxed{\frac{K}{Y} = \frac{\alpha}{B-1+\delta}.}$$

Now use the aggregate resource constraint:

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t.$$

On the BGP,

$$K_{t+1} = (1+g)K_t.$$

Substitute:

$$C_t + (1+g)K_t = Y_t + (1-\delta)K_t.$$

Rearrange:

$$C_t = Y_t - [g+\delta]\,K_t.$$

Divide by Y_t:

$$\frac{C}{Y} = 1 - (g+\delta)\frac{K}{Y}.$$

Using

$$\frac{K}{Y} = \frac{\alpha}{B - 1 + \delta},$$

we get

$$\boxed{\frac{C}{Y} = 1 - (g + \delta)\frac{\alpha}{B - 1 + \delta}.}$$

The condition for positive consumption is

$$\frac{C}{Y} > 0.$$

Thus,

$$1 - (g + \delta)\frac{K}{Y} > 0.$$

Equivalently,

$$\boxed{(g + \delta)\frac{K}{Y} < 1.}$$

Substituting the expression for K/Y:

$$\boxed{(g + \delta)\frac{\alpha}{B - 1 + \delta} < 1.}$$

Marking guide: output-capital ratio, 4; capital-output ratio, 2; consumption-output ratio, 3; positivity condition, 1.

Part (e) **[7 marks]**

The parameters are

$$\beta = 0.96, \qquad \theta = 2, \qquad \alpha = \frac{1}{3}, \qquad \delta = 0.05.$$

First consider

$$B_0 = 1.05.$$

The gross growth rate is

$$1 + g_0 = (\beta B_0)^{1/\theta} = (0.96 \times 1.05)^{1/2}.$$

Since

$$0.96 \times 1.05 = 1.008,$$

$$1 + g_0 = \sqrt{1.008} \approx 1.00399.$$

Hence,

$$\boxed{g_0 \approx 0.00399.}$$

The time allocation is

$$\varphi_0^* = 1 - \frac{1 + g_0}{B_0} = 1 - \frac{1.00399}{1.05} \approx 0.04382.$$

Thus,

$$\boxed{\varphi_0^* \approx 0.0438.}$$

The output-capital ratio is

$$\frac{Y}{K} = \frac{B_0 - 1 + \delta}{\alpha} = \frac{1.05 - 1 + 0.05}{1/3} = 0.3000.$$

Therefore,

$$\boxed{\frac{Y}{K} = 0.3000, \qquad \frac{K}{Y} = 3.3333.}$$

The consumption-output ratio is

$$\frac{C}{Y} = 1 - (g_0 + \delta)\frac{K}{Y}.$$

Thus,

$$\frac{C}{Y} = 1 - (0.00399 + 0.05)(3.3333) \approx 0.8200.$$

So,

$$\boxed{\frac{C}{Y} \approx 0.8200.}$$

Now consider

$$B_1 = 1.10.$$

The gross growth rate is

$$1 + g_1 = (0.96 \times 1.10)^{1/2}.$$

Since

$$0.96 \times 1.10 = 1.056,$$

$$1 + g_1 = \sqrt{1.056} \approx 1.02762.$$

Hence,

$$\boxed{g_1 \approx 0.02762.}$$

The time allocation is

$$\varphi_1^* = 1 - \frac{1.02762}{1.10} \approx 0.06580.$$

Thus,

$$\boxed{\varphi_1^* \approx 0.0658.}$$

The output-capital ratio is

$$\frac{Y}{K} = \frac{1.10 - 1 + 0.05}{1/3} = 0.4500.$$

Hence,

$$\boxed{\frac{Y}{K} = 0.4500, \qquad \frac{K}{Y} = 2.2222.}$$

The consumption-output ratio is

$$\frac{C}{Y} = 1 - (0.02762 + 0.05)(2.2222) \approx 0.8275.$$

So,

$$\boxed{\frac{C}{Y} \approx 0.8275.}$$

The comparison is:

	$B_0 = 1.05$	$B_1 = 1.10$
g	0.00399	0.02762
φ^*	0.04382	0.06580
Y/K	0.3000	0.4500
K/Y	3.3333	2.2222
C/Y	0.8200	0.8275

A higher return to education raises the endogenous growth rate because human capital accumulation becomes more productive. It also raises the optimal fraction of time spent working, φ^*, but this does not mean less education in growth terms: because B is higher, a given amount of education time produces more human capital growth. The economy grows faster, has a higher output-capital ratio, and can sustain a slightly higher consumption-output ratio in this numerical example.

Marking guide: computations for B_0, 3; computations for B_1, 3; interpretation, 1.